The Effect of Exogenous Inputs and Defiant Agents in Opinion Dynamic Systems with Local and Global Interactions

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Abstract

Most of the conventional models for opinion dynamics mainly account for a fully local influence, where myopic agents decide their actions after they interact with other agents that are adjacent to them. For example, in the case of social interactions, this includes family, friends, and other strong social ties. The model proposed in this contribution, embodies a global influence as well where, by global, we mean that each node also observes a sample of the average behavior of the entire population (in the social example, people observe other people on the streets, subway, and other social venues). We consider a case where nodes have dichotomous states (examples include elections with two major parties, whether or not to adopt a new technology or product, and any yes/no opinion such as in voting on a referendum). The dynamics of states on a network with arbitrary degree distribution are studied. For a given initial condition, we find the probability to reach consensus on each state and the expected time reach to consensus. The effect of an exogenous bias on the average orientation of the system is investigated, to model mass media. To do so, we add an external field to the model that favors one of the states over the other. This field interferes with the regular decision process of each node and creates a constant probability to lean towards one of the states. We solve for the average state of the system as a function of time for given initial conditions. Then anti-conformists (stubborn nodes who never revise their states) are added to the network, in an effort to circumvent the external bias. We find necessary conditions on the number of these defiant nodes required to cancel the effect of the external bias. Our analysis is based on a mean field approximation of the agent opinions.

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I. Introduction

In many complex network problems originating from the social and economic sciences, the aim is to study and understand the collective network behavior under particular microdynamics. Opinion dynamics models are a subset of these efforts; agent-based models are studied to elucidate the macro-behavior, given a prescribed interaction scheme. In opinion dynamics models each node is endowed with a state, which adapts and evolves based on observations of the states of other nodes in the network. The state is usually a number, which can be considered continuous [1] or discrete [2]. Depending on the problem, nodes may represent, e.g., people [3], firms [4], or countries [5]. At each time step, each agent observes the states of other agents with which it interacts, and then the agents update their states based on these observations.

In this paper we study the effects of different interaction phenomenon on the behavior of opinion dynamics models running over networks. Each agent has a binary state ± 1. Examples of cases where such a model may be applicable are: elections with two major parties [6], [7], deciding whether or not to adopt a new technology [8], [9], whether or not to watch a new movie [10], whether or not to follow a new fashion trend [11], and any other yes/no proposition (go to war or not, referendums, etc.).

To model the social factors that effect each agent's state, we initially consider two different sources of influence:

- Local influence: Each agent is influenced by its direct neighbors in the network (e.g., family, close friends and relatives, and other strong social ties).

- Global influence: Each agent is influenced indirectly by the average opinion of the population (e.g., by observing people on the streets, classes, subway, cinemas, and other social places). For example, one takes a sample of the popularity of a fashion trend or a new technological product (such as smart phones or laptops) by observing other people in the society. The existence of a strong tie is not necessary in this case. Another major example of the global type of influence is the internet.

We model both of these sources of influence. At each time step, each node observes the fraction of its neighboring nodes who have each of the two states. These fractions constitute the local part of the social influence, as will be explained in the next section. Each node also observes the fraction of all the nodes who adopt each of the two states. The global part of the social influence will be based on these fractions.

We find an expression for the evolution of the average density of each state over time (i.e. the fraction of nodes who adopt $s = +1$ and who adopt $s = -1$) as a function of the initial conditions. We consider a network with arbitrary degree distribution and conduct our analysis under the mean field approximation.

We also find the probability that all nodes will end up having the same state (i.e., reach consensus), and we calculate the expected times to reach consensus on either of the two states. The conventional (or "pure") voter model [12] is a limiting case of our setup, where the global influence is zero, and our results can be seen as generalizing those of [13] for the pure voter model.

Next, we consider the effect of an exogenous field that influences all agents, biasing them towards $+1$. For example, this could model the effects of mass media. We find an expression for the evolution of the expected state over time, as a function of the initial conditions. This expression reveals the rate of convergence of all nodes to the state $s = +1$, depending on the magnitude of the external field.

Finally, we incorporate and study the effect of anti-conformists: stubborn nodes who fix their opinion at -1, opposing the external field. In this case, there is never a consensus; instead, the network reaches an equilibrium with a certain fraction of agents taking each side. For this model we determine the fraction of stubborn nodes required to sway the majority opinion to -1 for an exogenous field of a particular strength.

A. Related Work

There is a growing literature on opinion dynamics and related models, and we review the most related work here. Different opinion dynamics models have different interaction processes and different updating schemes, which are contingent on the specificities of the problem. For example, one can choose random blocks of adjacent nodes who enforce their opinion on neighbors under certain conditions [14]–[16]. In the so-called *voter model*, at each timestep, a node is randomly chosen, who then copies the state of a randomly chosen neighbor [2], [7], [12], [13], [17]–[19]. Conversely, in the *invasion process*, the randomly chosen node imposes its state on a randomly chosen neighbor [20].

Learning and trust have been incorporated in the models, where each node has an estimate of how reliable its observations might be [3], [21]–[25]. In some models, nodes only interact with neighbors whose opinions are close to their own [26]–[29]. In [30], agents have inertia in the sense that the longer they have had a particular state, the less probable it gets for them to deviate from it.

In [31]–[33], the effect of stubborn agents (a.k.a., zealots) are studied. These agents commit to a certain state and never vary. In [31] the effect of a single zealot is studied, and it is shown that in one and two dimensions the system reaches consensus over the state enforced by the zealot, but this is not true in higher dimensions. In [32] the voter model with arbitrary number of zealots is examined on the complete graph, and lattice graphs in one and two dimensions, and it is shown that the magnetization has Normal distribution whose characteristics only depend on the number of zealots, not on the entire population. The

upshot is that a few number of zealots can prevent consensus, regardless of the population size. In [33], the expected values of the average state for arbitrary number of zealots on an arbitrary graph is found for the pure voter model, and it is shown that consensus is prevented as long as stubborn nodes exist. Also, the steady state of the system is shown to be independent of the initial states of the non-stubborn nodes.

In [13] the pure voter model on heterogeneous graphs is solved under the mean-field assumption. This means that nodes with equal degrees are considered to be indistinguishable in dynamics, and the connection probability of each pair of nodes is proportional to their degrees. They find the probability to reach consensus on each of the states, and also the expected time to reach consensus. In [34], intrinsic "flip" rates are heterogeneous, so that some nodes adopt new opinions more frequently than others.

B. Paper Organization

The rest of this paper is organized as follows. The basic model, incorporating both local and global influence, is formally defined and studied in Section II. Then, Section III studies the effect of an exogenous field. Section IV introduces anti-conformists to the model. Finally, we conclude in Section V. Throughout the paper, theoretical predictions are accompanied by the simulation results.

II. DYNAMICS IN THE ABSENCE OF EXOGENOUS BIAS

A. Network Model and Basic Dynamics

We consider a network of N nodes[1]. The network structure is defined through the neighborhoods of each node. For node x, let N_x denote the neighbors of node x (i.e., the set of nodes adjacent to x), and let $z_x = |N_x|$ be the degree of node x (the number of its neighbors). The underlying network is assumed to be undirected: for two nodes x and y, we have $y \in N_x$ if and only if $x \in N_y$. The network is also assumed to be connected: for any two agents x and y, there is a sequence of nodes $x_0 = x, x_1, \ldots, x_\ell = y$ such that $x_k \in N_{x_{k-1}}$ for each $k = 1, \ldots, \ell$. We will be particularly interested in random network models, such as Erdos-Renyi graphs [35], [36], preferential attachment graphs [37], and random recursive trees (as in [38], [39], with $A_k = 1$).

Each node has a binary-valued state $s_x(t) \in \{-1, +1\}$. Nodes have initial opinions $s_x(0)$ at time $t = 0$, and time progresses forwards. At each time step, one node is picked randomly, and it updates its state. In general, the update will involve the node either keeping the same state, or flipping to the other

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[1]Throughout, we use the terms "agent" and "node" interchangeably.

state. In order to avoid burdensome notation below, we will omit the dependence on time when it is clear from the context. Let us denote the state of the system by a vector $\vec{s} = (s_1, s_2, \ldots, s_N)$. The probability that a node x flips its state at time t is given by

$$P\{s_x(t + \delta t) = -s_x(t)|s_x(t)\} = w_x(\vec{s}(t)),$$

where

$$w_x(\vec{s}) = \frac{\theta}{2}\left(1 - \frac{s_x}{z_x}\sum_{y \in N_x} s_y\right) + \frac{1 - \theta}{2}\left(1 - \frac{s_x}{N}\sum_{\text{all } y} s_y\right). \tag{1}$$

The first term, with weight $\theta \in [0, 1]$, captures the local influence. It is equal to the fraction of neighbors who oppose node x. The second part is the global influence, with weight $1 - \theta$. It equals the fraction of the entire population that disagree with node x. Here θ is a tradeoff parameter between the influence of close social ties and the influence of the society. Node x, upon being selected, flips its state with probability w_x. Hence, if all nodes disagree with x then x flips its state with probability 1. On the other hand, if all nodes agree with x then x does not flip its state with probability 1.

B. A Conservation Law and Consensus Probabilities

Now we focus on finding the probability that nodes reach consensus on $s = +1$ and on $s = -1$. Note that when the system is not unanimous, at each timestep there are nonzero probabilities to move towards the states $+1$ or -1 since, as long as consensus has not been reached, there is at least one node with nonzero flipping probability which, by construction, has nonzero probability for being picked to update its state. Hence the only absorbing states of the system are those where either all nodes have $+1$ or -1.

We solve the model in continuous-time approximation, examining the system as $\delta t \to 0$, in which case for any quantity $f(t)$ we have $\frac{f(t+\delta t) - f(t)}{\delta t} \to \dot{f} = \frac{\partial f}{\partial t}$. Let c_x denote the expected value of s_x. The evolution of c_x is given by

$$\dot{c}_x = -c_x + \frac{\theta}{z_x}\sum_{y \in N_x} c_y + \left(\frac{1 - \theta}{N}\sum_{\text{all } y} c_y\right). \tag{2}$$

Let n_k denote the fraction of nodes with degree k, and let $\bar{z} = \frac{1}{N}\sum_x z_x = \sum_k k n_k$ denote the average degree of the network. Also define $\rho_k(t)$ to be the fraction of nodes with degree k which have $s_x(t) = +1$. Consider the quantity $\mu(t) \stackrel{\text{def}}{=} \sum_k k n_k \rho_k$, which can be interpreted as the expected value of the quantity $k\rho_k$ for a graph with degree distribution $\{n_k\}$.

Under the assumption that the degrees of adjacent nodes are uncorrelated, the probability that two nodes are connected is proportional to the product of their degrees; i.e.,

$$P\{y \in N_x|z_y = k\} = \frac{k z_x}{N\bar{z}}, \tag{3}$$

and $P\{y \in N_x\} = z_y/N$. Also recall that $n_k = P\{z_y = k\}$. Then by Bayes' rule, we have

$$P\{z_y = k | y \in N_x\} = \frac{\frac{kz_x}{N\bar{z}} \cdot n_k}{\frac{z_x}{N}} = \frac{kn_k}{\bar{z}}.$$

The assumption that the degrees of adjacent nodes are uncorrelated is justified in a number of scenarios. For example, this is the case in the classical Erdos-Renyi random graph model [35], [36]. Social networks have also been observed to have preferential attachment properties, and in the well-known scale-free network model of Barabasi and Albert [37] edges are no longer independent since new edges connect to high-degree nodes with higher probability. Nonetheless, in Section II-D we justify that (3) still holds true for scale-free graphs.

We will also make a mean-field assumption and approximate that, for a neighbor $y \in N_x$ with degree $z_y = k$, the expected state c_y is equal to $2\rho_k - 1$, the average state of all nodes with degree k. Under these approximations, we have

$$\sum_{y \in N_x} c_y \approx z_x \sum_k (2\rho_k - 1) \cdot \frac{kn_k}{\bar{z}}$$

$$= z_x \left(\sum_k \frac{2k\rho_k n_k}{\bar{z}} - \sum_k \frac{kn_k}{\bar{z}} \right)$$

$$= z_x \left(\frac{2\mu}{\bar{z}} - 1 \right). \tag{4}$$

Let $m(t) = \frac{1}{N} \sum_x s_x(t)$ denote the average state at time t. Then combining (2) and (4) gives the following differential equation for $m(t)$:

$$\dot{m}(t) = -m(t) + \theta \left(\frac{2\mu}{\bar{z}} - 1 \right) + (1 - \theta)m(t). \tag{5}$$

Multiplying (2) by z_x and summing over all nodes gives the following expression for the evolution of $\mu(t)$:

$$\dot{\mu}(t) = \left[\frac{\bar{z}}{2}(1 - \theta) \right] m(t) - (1 - \theta)\mu(t) + \left(\frac{\bar{z}}{2}(1 - \theta) \right). \tag{6}$$

Comparing (5) with (6), observe that

$$\dot{m}(t) \left[\frac{\bar{z}}{2}(1 - \theta) \right] + \theta \dot{\mu}(t) = 0. \tag{7}$$

Thus, we arrive at the following conservation law:

$$\psi(t) \stackrel{\text{def}}{=} m(t) \left[\frac{\bar{z}}{2}(1 - \theta) \right] + \theta \mu(t) = \text{const.} \tag{8}$$

For the pure voter model (i.e., with only local influences), we have $\theta = 1$ and $\psi = \mu$ is conserved, which matches the findings of [19].

Now this conservation law can be used to directly determine the probability that the network reaches consensus on a particular state, given the initial opinions at each node. Note that ψ equals $\frac{\bar{z}}{2}(1 + \theta)$ for the case where $s_x = +1$ for all x, and $\psi = \frac{-\bar{z}}{2}(1 - \theta)$ when $s_x = -1$ for all x. For brevity let us define:

$$
\begin{cases}
P^u \overset{\text{def}}{=} P\{m(\infty) = 1\} \\
P^d \overset{\text{def}}{=} P\{m(\infty) = -1\}
\end{cases}
\tag{9}
$$

Using the fact that

$$
\psi(0) = \lim_{t \to \infty} \psi(t) = P^u \cdot \frac{\bar{z}}{2}(1 + \theta) + (1 - P^u) \cdot \frac{-\bar{z}}{2}(1 - \theta),
$$

the probability of reaching consensus on $s = +1$ is easily found to be

$$
P^u = \frac{\psi(0)}{\bar{z}} + \frac{1 - \theta}{2}.
\tag{10}
$$

Similarly, we find

$$
P^d = \frac{1 + \theta}{2} - \frac{\psi(0)}{\bar{z}}.
\tag{11}
$$

In terms of the initial states, we can rewrite (10) as

$$
P^u = \frac{1}{2} + \frac{1 - \theta}{2N} \sum_x s_x(0) + \frac{\theta}{2N\bar{z}} \sum_x z_x s_x(0),
\tag{12}
$$

and (11) also can be rewritten as

$$
P^d = \frac{1}{2} - \frac{1 - \theta}{2N} \sum_x s_x(0) - \frac{\theta}{2N\bar{z}} \sum_x z_x s_x(0).
\tag{13}
$$

Figure 1 illustrates the probabilities P^u and P^d for reaching consensus on ± 1 as a function of initial fraction of $+1$ nodes for opinion dynamics run on Barabasi-Albert scale free graphs with 400 nodes. The initial opinions at each node are drawn i.i.d., with the initial density of state $+1$ varying between 0.05 and 0.95, and the local-global tradeoff parameter θ drawn uniformly between 0 and 1. As expected, the probability P^u to reach consensus on $+1$ is higher when more nodes initially have state $+1$, and this probability is linear in the initial density of nodes with $+1$.

Next, we can solve for the dynamics of the average state, $m(t)$. Equations (5) and (6) are a standard system of linear differential equations (see [40]–[42] for example). The solution for $\mu(t)$ is

$$
\mu(t) = \left[\psi_0 + \frac{\bar{z}}{2}(1 - \theta) \right] + \left[\mu_0 - \psi_0 - \frac{\bar{z}}{2}(1 - \theta) \right] e^{-t},
\tag{14}
$$

and for $m(t)$ we have

$$
m(t) = \left[\frac{2}{\bar{z}} \psi_0 - \theta \right] + \left[m_0 - \frac{2}{\bar{z}} \psi_0 + \theta \right] e^{-t}.
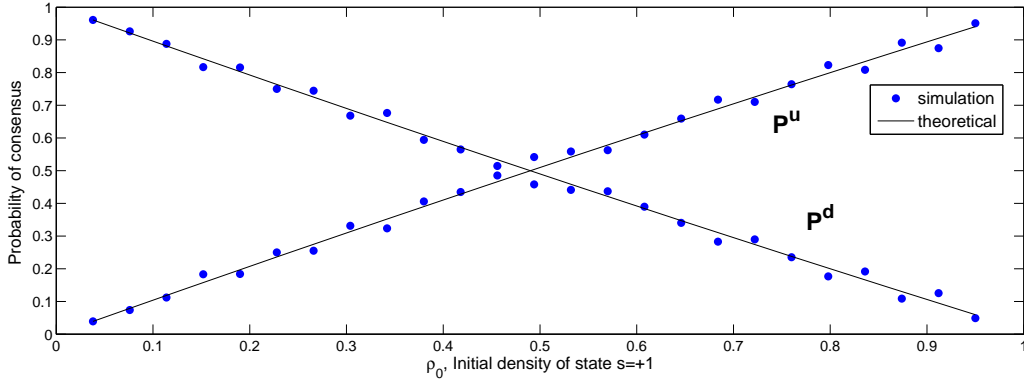\tag{15}
$$

Fig. 1: The expected probability of reaching consensus on the state $+1$ for different initial conditions for the fraction of nodes with $s = +1$, on a network of 400 nodes, for different values of θ between zero and one (uniform). The results are averaged over 1000 Monte Carlo simulations. The underlying graph is a Barabasi-Albert scale-free graph [37] with $m = 2$.

For both $m(t)$ and $\mu(t)$ there is a steady-state value and there is a transient exponentially-decaying part. Figures 2 and 3 are depictions of the theoretical prediction and simulation results for the steady-state values.
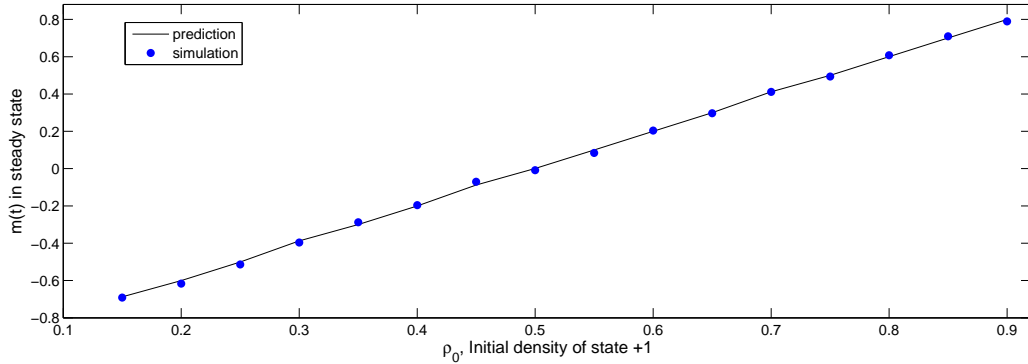

Fig. 2: Steady state values of $m(t)$ as a function as $\rho(0)$ for different values of $\rho(0)$ between $0.1, 0.9$ (uniform) and the predicted value of equation (7) with $\theta = 0.7$. The underlying graph is a random recursive tree (as in [38], with $A_k = 1$) with 200 nodes. Results are averaged over 100 Monte Carlo trials.

Note that for $\theta = 0$ (i.e., purely global influence, which is equivalent to a complete graph), equation
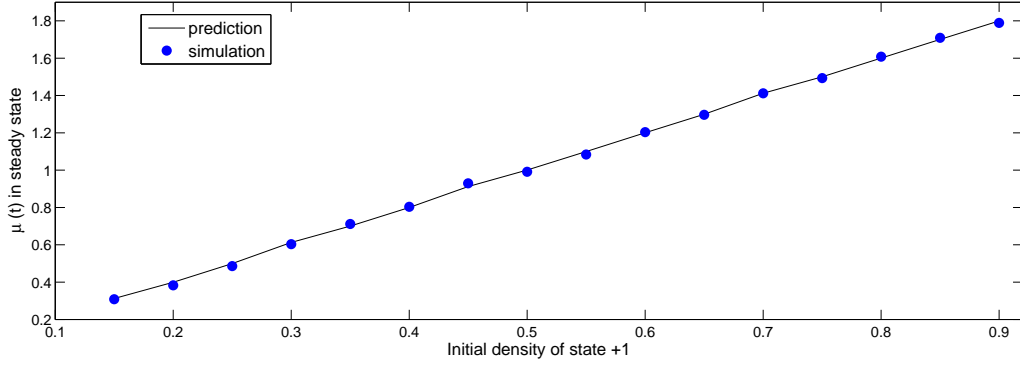
Fig. 3: Steady state values of $\mu(t)$ as a function as $\rho(0)$ for different values of $\rho(0)$ between $0.1, 0.9$ (uniform) and the predicted value of equation (7) with $\theta = 0.7$. The underlying graph is a random recursive tree (as in [38], with $A_k = 1$) with 200 nodes. Results are averaged over 100 Monte Carlo trials.

(8) implies that m is conserved. Also for $\theta = 1$ the conserved quantity is μ. Both of these polar cases agree with the results in [13].

C. Time to Consensus

Now let us focus on the expected time required for all nodes to end up with the same state, and expected consensus time on $s = +1$ and $s = -1$ individually. Recall that n_k is the fraction of nodes with degree k; thus $N n_k$ is the number of nodes with degree k. Of the nodes with degree k, if one changes state from -1 to $+1$, then note that ρ_k changes by $1/(N n_k)$. Let us define $\delta \rho_k = \frac{1}{N n_k}$, and define

$$\begin{cases} P_k^+ \stackrel{\text{def}}{=} P\{\rho_k \to \rho_k + \delta \rho_k\} \\ P_k^- \stackrel{\text{def}}{=} P\{\rho_k \to \rho_k - \delta \rho_k\} \end{cases} . \tag{16}$$

For the first one we have

$$P_k^+ = n_k(1 - \rho_k)\left[\frac{1}{2} + \frac{\theta}{2}\left(\frac{2\mu}{\bar{z}} - 1\right) + \frac{1 - \theta}{2}m\right] = n_k(1 - \rho_k)\left(\frac{1 - \theta}{2} + \frac{\theta}{\bar{z}}\mu + \frac{1 - \theta}{2}m\right), \tag{17}$$

which implies that a degree-k node with state -1 is pickedand it flips its state at this step. Similarly,

$$P_k^- = n_k\rho_k\left[\frac{1}{2} - \frac{\theta}{2}\left(\frac{2\mu}{\bar{z}} - 1\right) - \frac{1 - \theta}{2}m\right] = n_k\rho_k\left(\frac{1 + \theta}{2} - \frac{\theta}{\bar{z}}\mu - \frac{1 - \theta}{2}m\right). \tag{18}$$

Let us denote by $T(\vec{\rho})$ the expected consensus time when the densities for population of nodes with various degrees are $\vec{\rho} = (\rho_1, \rho_2, \rho_3, \ldots)$. Note that $T(\vec{\rho})$ is the expected time to reach consensus on either

state. We will find the expected time to reach consensus on individual states $+1$ or -1 afterwards. The expected consensus time satisfies the following recurrence relation:

$$T(\vec{\rho}) = \delta t + \sum_k \left[T(\vec{\rho} + \delta\rho_k \hat{k})P_k^+ + T(\vec{\rho} - \delta\rho_k \hat{k})P_k^- \right] + \left[1 - \sum_k (P_k^+ + P_k^-) \right] T(\vec{\rho}). \tag{19}$$

The first term is the time increment due to an update occurring. The second term accounts for the expected time to consensus, given that the selected node has changed its state. The variable k in the sum runs over all possible degrees in the network. The last term is the expected time to consensus, given that the selected node has made no change in its state.

Let us normalize time units so that $\delta t = \frac{1}{N}$. Define the constants

$$\begin{cases} \lambda \overset{\text{def}}{=} \frac{1-\theta}{2} + \frac{\psi}{\bar{z}} \\ \alpha \overset{\text{def}}{=} \theta - \frac{2\psi}{\bar{z}} \end{cases} . \tag{20}$$

Since ψ is conserved, we will drop the argument t.

Let us denote 8 by $\partial_k T(\vec{\rho})$ for brevity. In the continuous approximation (as $N \to \infty$), equation (19) becomes

$$\sum_k (\lambda - \rho_k)\partial_k T(\vec{\rho}) + \sum_k \frac{\delta\rho_k}{2} \left[\lambda + \alpha\rho_k \right] \partial_k^2 T(\vec{\rho}) = -1. \tag{21}$$

To simplify this equation we note that the change in $\bar{\rho}_k$, the expected value of ρ_k. We find

$$\frac{d}{dt}\bar{\rho}_k = P_k^+ - P_k^- = n_k(\lambda - \bar{\rho}_k) \implies \bar{\rho}_k(t) = \lambda + (\rho_k(0) - \lambda)e^{-n_k t}, \tag{22}$$

which means that the deviation of ρ_k from the central value λ decays exponentially, and thus the expected density $\bar{\rho}_k$ rapidly approaches λ for all k. It is observed in the simulations that this phase is very short compared to the rest of the process (see Table I). Also Figure 4 shows samples of the time behavior of ρ_k in simulation.

Using (22), the first term in (21) vanishes, and (21) simplifies to:

$$\sum_k \frac{\delta\rho_k}{2} \left[\lambda + \alpha\rho_k \right] \partial_k^2 T(\vec{\rho}) = -1, \tag{23}$$

or equivalently,

$$\sum_k \left(\frac{1-\theta}{2} + \frac{\psi}{\bar{z}} \right) \left(\frac{1+\theta}{2} - \frac{\psi}{\bar{z}} \right) \frac{n_k}{N} \left[\theta k + \bar{z}(1-\theta) \right]^2 \partial_\psi^2 T = -1. \tag{24}$$

Note that when every node has $s = +1$ then the time to consensus is zero. In this case we have $m = 1$ and $\mu = \bar{z}$. Similarly, the time to consensus is also zero when all nodes have $s = -1$, in which case

N	T	T_λ	$\frac{T_\lambda}{T}$
100	61	3	0.049
200	149	3	0.020
300	221	5	0.023
400	317	6	0.019
500	402	8	0.020
600	445	8	0.018
700	496	9	0.018
800	549	12	0.022
900	638	14	0.022
1000	755	15	0.020

TABLE I: Comparing the average time to reach consensus (denoted by T), and the short initial phase required for nodes of each degree to reach $\rho_k = \lambda(1 \pm 0.03)$ (denoted by T_λ). The value of θ is chosen randomly in the interval of $(0.1, 0.4)$ for each simulation. T_λ is averaged over all degrees. The underlying graph is a scale-free Barabasi-Albert graph as in [37], with $m = 2$. The quantities are averaged over 100 Monte Carlo simulations for each case. The initial distribution of states is generated randomly for each simulation.

$m = -1$ and $\mu = 0$. These observations give us the boundary conditions required to solve the differential equation:

$$\begin{cases} T\left(\psi = \frac{\bar{z}}{2}(1 + \theta)\right) = 0 \\ T\left(\psi = \frac{-\bar{z}}{2}(1 - \theta)\right) = 0 \end{cases}. \tag{25}$$

Let us define the constant

$$A \overset{\text{def}}{=} -\frac{N}{\theta^2 \frac{\langle z^2 \rangle}{\bar{z}^2} + 1 - \theta^2}, \tag{26}$$

where $\langle z^2 \rangle = \frac{1}{N} \sum_x z_x^2$ is the average over all nodes of the degree squared. Then equation (23) can be written as

$$A \left(\frac{1 - \theta}{2} + \frac{\psi}{\bar{z}}\right) \left(\frac{1 + \theta}{2} - \frac{\psi}{\bar{z}}\right) \partial_\psi^2 T = 1. \tag{27}$$

The expected consensus time is given by the solution of (23) with the boundary conditions (25). The solution is readily obtained by integrating twice:

$$T = A \left\{ \left(\frac{1 - \theta}{2} + \frac{\psi}{\bar{z}}\right) \left[\ln\left(\frac{1 - \theta}{2} + \frac{\psi}{\bar{z}}\right)\right] + \left(\frac{1 + \theta}{2} - \frac{\psi}{\bar{z}}\right) \left[\ln\left(\frac{1 + \theta}{2} - \frac{\psi}{\bar{z}}\right)\right] \right\}. \tag{28}$$
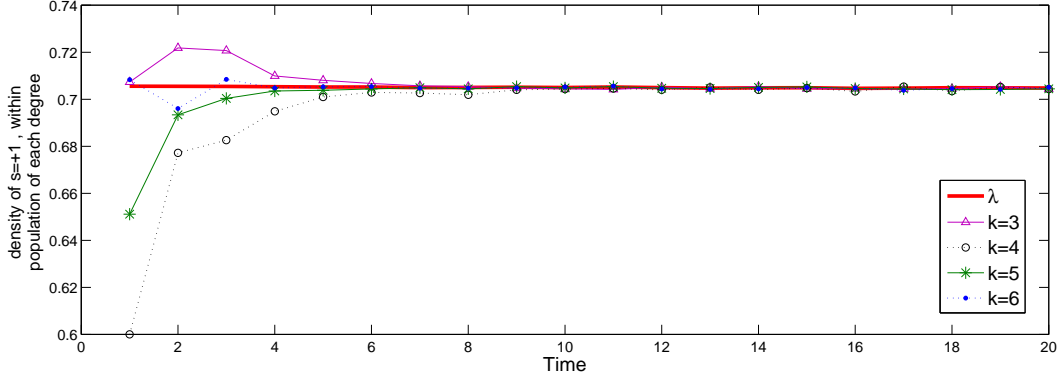
Fig. 4: Simulation results compared to what 22 predicts. Results are averaged over 1500 Monte Carlo simulations (with identical initial conditions), for a Barabasi-Albert graph [37] with $N = 500$ nodes and $m = 2$. The initial condition was $\rho = 0.7$, with nodes selected randomly regardless of degrees. The value of θ is 0.7.

We can use P^u and P^d to get a shorter expression for T:

$$T = A[P^u \ln P^u + P^d \ln P^d], \tag{29}$$

or equivalently,

$$T = A\left[P^u \ln P^u + (1 - P^u)\ln(1 - P^u)\right]. \tag{30}$$

Hence, the time to consensus depends on an entropy-like quantity involving the probability of reaching consensus on either state. Note that $A < 0$. Thus, this expression confirms with the intuition that the expected time to reach consensus is largest when the probability of reaching consensus on either of the states ± 1 is equal, given the initial conditions.

The theoretical prediction for T is validated in simulation in Figure 5. Also Figure 6 is a depiction of consensus time as the population size changes, for a Barabasi-Albert network and random recursive tree (as defined in [38], with $A_k = 1$).

Now let us examine two polar cases, $\theta = 1$ and $\theta = 0$. These polar cases were previously studied in [19] and [13]. When $\theta = 1$, we get the conventional voter model as discussed in [13]. From (12) and (26) we have

$$\begin{cases} A = \dfrac{-N}{\frac{\langle z^2 \rangle}{\bar{z}^2}} = -N\dfrac{\bar{z}^2}{\langle z^2 \rangle} \\[4mm] P^u = \dfrac{\mu}{\bar{z}}, \end{cases} \tag{31}$$

which gives

$$T = -N \frac{\bar{z}^2}{\langle z^2 \rangle} \left[\left(1 - \frac{\mu}{\bar{z}}\right) \ln \left(1 - \frac{\mu}{\bar{z}}\right) + \left(\frac{\mu}{\bar{z}}\right) \ln \left(\frac{\mu}{\bar{z}}\right) \right]. \tag{32}$$

This corresponds to Equation (13) of [13].

For the other case, $\theta = 0$, we have:

$$\begin{cases} A = -N \\ P^u = \rho, \end{cases} \tag{33}$$

which gives

$$T = -N \left[(1 - \rho) \ln(1 - \rho) + \rho \ln \rho \right], \tag{34}$$

This matches Equation (25) in [19]. Thus, the expressions derived in this expression can be viewed as generalizing the previous work [13], [19] to interpolate between the cases where agents experience either purely local or purely global influence.
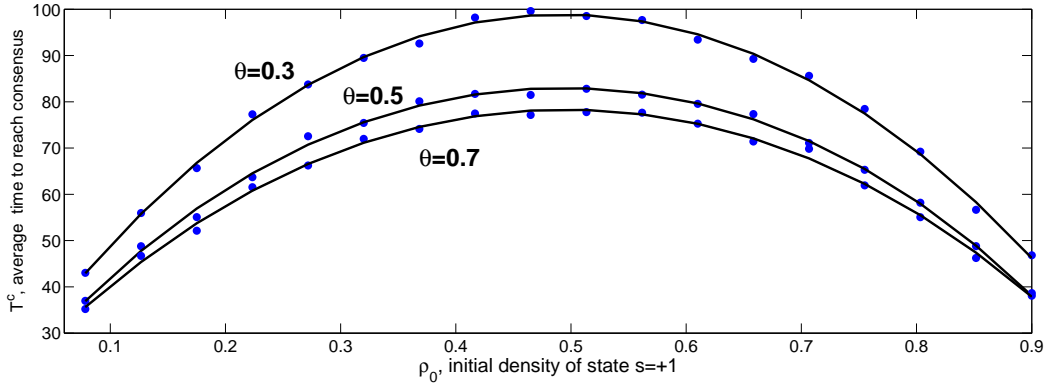

Fig. 5: The expected time to reach consensus for different initial conditions for the fraction of nodes with $s = +1$, on a network of 100 nodes, for a Barabasi-Albert graph [37] with $m = 2$, for different values of θ.

The analysis above found an expression for T, the expected time to reach consensus on any state. Next, we refine our analysis by characterizing the time to consensus given that consensus is reached on a particular state. Denote by T^u the time to reach consensus given that consensus is reached on $+1$, and let T^d be defined analogously for consensus on -1. Similar to (19), the recurrence relation for T^u
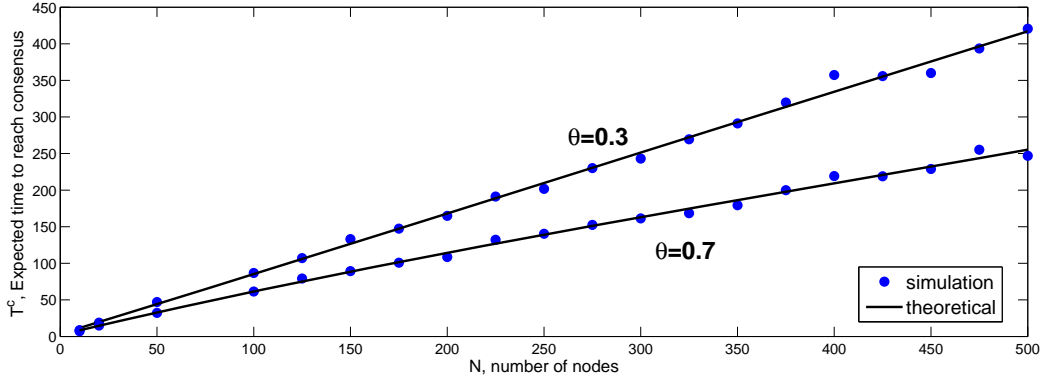
Fig. 6: Consensus time as a function of N for a Barabasi-Albert graph [37] with $m = 2$. The number of nodes is 200 and results are averaged over 500 Monte Carlo simulations.

becomes:

$$P^u(\vec{\rho})T^u(\vec{\rho}) = P^u(\vec{\rho})\delta t + \left[1 - \sum_k (P_k^+ + P_k^-)\right] P^u(\vec{\rho})T^u(\vec{\rho})$$

$$+ \sum_k \left[P^u(\vec{\rho} + \delta\rho_k \hat{k})T^u(\vec{\rho} + \delta\rho_k \hat{k})P_k^+ + P^u(\vec{\rho} - \delta\rho_k \hat{k})T^u(\vec{\rho} - \delta\rho_k \hat{k})P_k^-\right]. \quad (35)$$

Following a similar line of reasoning that led to (23), we arrive at the following differential equation:

$$\sum_k \left(\frac{1-\theta}{2} + \frac{\psi}{\bar{z}}\right)\left(\frac{1+\theta}{2} - \frac{\psi}{\bar{z}}\right)\frac{n_k}{N}\left[\theta k + \frac{\bar{z}}{2}(1-\theta)\right]^2 \partial_\psi^2 P^u T^u = -P^u. \quad (36)$$

Using (26), this further simplifies to:

$$A\left(\frac{1-\theta}{2} + \frac{\psi}{\bar{z}}\right)\left(\frac{1+\theta}{2} - \frac{\psi}{\bar{z}}\right)\partial_\psi^2\left[P^u T^u\right] = P^u \quad (37)$$

Now, note that the boundary conditions are different from (25). In the present case, if all nodes have $s = -1$ then the expected time to reach consensus on $s = +1$ goes to infinity. So we have:

$$\begin{cases} T^u\left(\psi = \frac{\bar{z}}{2}(1+\theta)\right) = 0 \\ T^u\left(\psi = \frac{\bar{z}}{2}(1-\theta)\right) \to \infty \end{cases}. \quad (38)$$

Using these boundary conditions, the solution to (37) is obtained by integrating twice:

$$T^u = A\frac{\frac{1+\theta}{2} - \frac{\psi}{\bar{z}}}{\frac{1-\theta}{2} + \frac{\psi}{\bar{z}}}\ln\left(\frac{1+\theta}{2} - \frac{\psi}{\bar{z}}\right). \quad (39)$$

Similarly, for the expected time to reach consensus given that consensus is reached on $s = -1$, we get:

$$T^d = A\frac{\frac{1-\theta}{2} + \frac{\psi}{\bar{z}}}{\frac{1+\theta}{2} - \frac{\psi}{\bar{z}}}\ln\left(\frac{1-\theta}{2} + \frac{\psi}{\bar{z}}\right). \quad (40)$$

Figures 7 and 8 compare the theoretical predictions of (39) and (40) to simulations.

We note that one can check that the following holds:

$$P^u T^u + P^d T^d = T. \tag{41}$$

This implies that T is the expected value of consensus, on a probability space of two events, consensus on $s = +1$ and consensus on $s = -1$ with respective probabilities P^u and P^d. Hence, one can also use this relation to extract T^d, once T^u and T are obtained, rather than solving differential equation congruent to (37) for T^d, which is obtained by replacing all P^us by P^ds, and T^u by T^d.
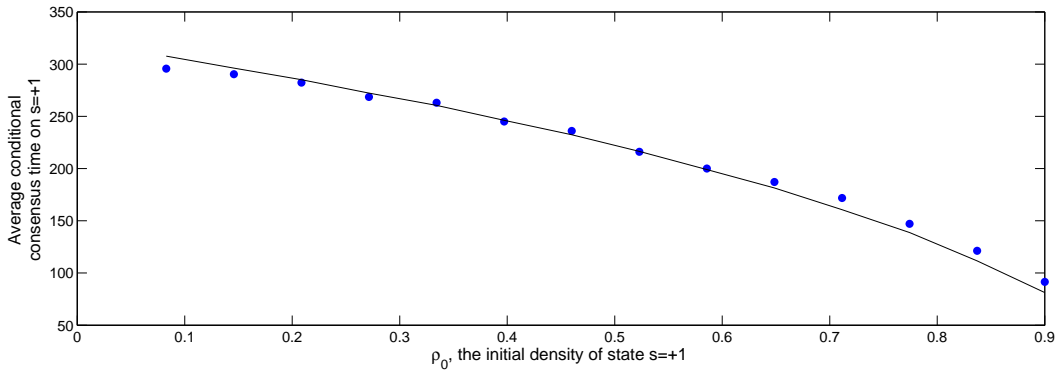

Fig. 7: Consensus time conditional upon consensus over $s = +1$. The underlying graph is a Barabasi-Albert [37] with $m = 2$ with 200 nodes. Results are averaged over 800 simulations.
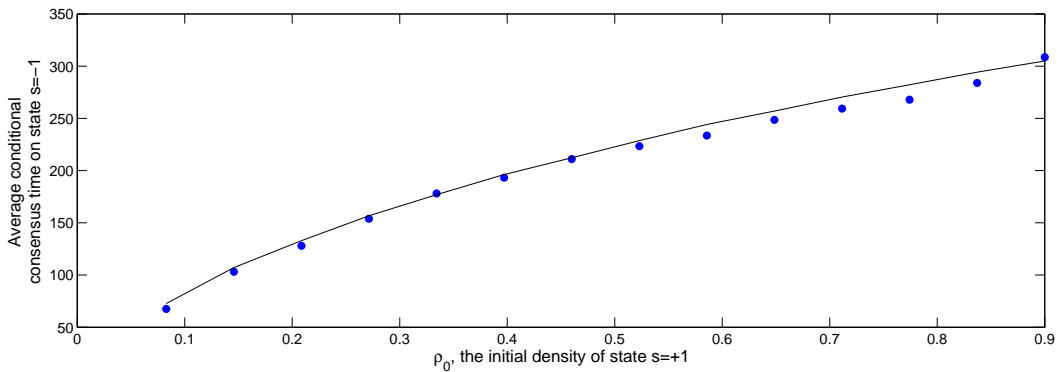

Fig. 8: Consensus time conditional upon consensus over $s = -1$. The underlying graph is a Barabasi-Albert [37] with $m = 2$ with 200 nodes. Results are averaged over 800 simulations.

D. Dynamics on the Barabasi-Albert Graph

Let us examine in the passing, the validity of (4) for the Barabasi-Albert model [37]. The Barabasi-Albert model is defined through a sequential algorithms for constructing a graph. In each step of the algorithm, one new node is introduced and is connected to m existing nodes. (Note that these steps are different than time steps in the opinion dynamics models studied above.) Let us say that node j is introduced at step j. Then the probability that node j forms a link with an existing node $i < j$ is

$$P(i \leftrightarrow j) = m \frac{z_i(j)}{\sum_{j' < j} z_{j'}(j)}, \tag{42}$$

where $z_i(j)$ is the degree of node i just before node j is added to the network.

This scheme is called *preferential attachment* to evoke the notion that existing nodes with higher degrees are more likely to receive new links. The denominator in (42) is equal to $j\bar{z}(j) = 2mj$ (i.e., twice the total number of links in the network when j is added), so we have

$$\frac{d}{dj} z_i(j) = \frac{z_i(j)}{2j} \quad \Longrightarrow \quad z_i(j) = m\sqrt{\frac{j}{i}}. \tag{43}$$

The probability of j linking to i becomes

$$P_t(i \leftrightarrow j) = \frac{z_i(t) z_j(t)}{2mt} \quad \Longrightarrow \quad P(i \leftrightarrow j) = \frac{z_i z_j}{N \bar{z}}. \tag{44}$$

This result is equivalent to saying that the ij-th element of the adjacency matrix is 1 with probability $\frac{z_i z_j}{N \bar{z}}$ and is zero otherwise. Using this result, we have

$$\sum_{y \in N_x} c_y = \sum_{\text{all } y} \frac{z_x z_y}{N \bar{z}} c_y = \sum_k \frac{z_x k}{N \bar{z}} N n_k (2\rho_k - 1) = \frac{z_x}{\bar{z}} \left[2 \sum_k k n_k \rho_k - \sum_k k n_k \right] = z_x \left[\frac{2\mu}{\bar{z}} - 1 \right], \tag{45}$$

which is consistent with (4).

III. Dynamics in the Presence of Exogenous Influence

Now we extend our model by adding an external bias that tries to align all the states towards +1. This external field models the effect of mass media. Examples include advertisements on TV, the internet, billboards, theatres, etc.

Each node assigns a weight to personal observations (which themselves are comprised of local and global interactions) and a weight to the external bias. This means that there is a constant probability that each node flips towards the external field. We denote this weight by $\gamma \in [0, 1]$ and the strength of the external field is denoted by $B \in (0, 1]$. A larger value of γ corresponds to less exposure to the media bias. It is clear that eventual consensus on +1 is inevitable since $B > 0$. Now we focus on how the dynamics of the system change when the field is introduced.

We begin by reexamining the conservation law derived in the previous section. The evolution of the expected state of a node at site x is

$$\dot{c}_x = \gamma \left[-c_x + \frac{\theta}{z_x} \sum_{y \in N_x} s_y + (1 - \theta)m \right] + (1 - \gamma)B. \tag{46}$$

Summing up this equation for all nodes gives

$$\dot{m} = (-\theta\gamma)m + \left(\frac{2\theta\gamma}{\bar{z}} \right) \mu - \theta\gamma + B(1 - \gamma). \tag{47}$$

Multiplying (46) by z_x and summing over all nodes yields

$$\dot{\mu} = \frac{\bar{z}\gamma}{2}(1 - \theta)m + (1 - \theta)\gamma\mu + \frac{\bar{z}}{2}\left[(1 - \theta)\gamma + (1 - \gamma)B \right]. \tag{48}$$

From (47) and (48) we see that $\psi(t)$ is no longer a conserved quantity; rather, it evolves as follows

$$\dot{\psi} = \frac{d}{dt} \left[\frac{\bar{z}}{2}(1 - \theta)m + \theta\mu \right] = \frac{\bar{z}}{2}(1 - \gamma)B, \tag{49}$$

so we have

$$\psi(t) = \psi(0) + (1 - \gamma)\frac{\bar{z}}{2}Bt. \tag{50}$$

Using this, we can express μ in (47) in terms of m and time, which yields a first order linear equation for $m(t)$. We get:

$$\dot{m} = -\gamma m + \frac{2\gamma}{\bar{z}}\psi_0 + \gamma(1 - \gamma)Bt + B(1 - \gamma). \tag{51}$$

Integrating this equation gives the average state over all nodes:

$$m(t) = B(1 - \gamma)t + \left[2\frac{\psi(0)}{\bar{z}} - \theta \right] + \left[m_0 - 2\frac{\psi(0)}{\bar{z}} + \theta \right] e^{-\gamma t}. \tag{52}$$

For $\mu(t)$ we plug (52) into (50) to get:

$$\mu(t) = \frac{\gamma\bar{z}}{2}B(1 - \gamma)t + \left[\psi(0) + \frac{\bar{z}}{2}(1 - \theta) \right] + \left[\mu_0 - \psi(0) - \frac{\bar{z}}{2}(1 - \theta) \right] e^{-\gamma t}. \tag{53}$$

Equation (22) in the presence of the external bias becomes

$$\frac{d}{dt}\bar{\rho}_k = P_k^+ - P_k^- = n_k \left\{ \underbrace{\gamma \left(\frac{1 - \theta}{2} \right) + B \left(\frac{1 - \gamma}{2} \right) + \frac{\gamma\psi(0)}{\bar{z}}}_{\overset{\text{def}}{=}a} - \gamma\rho_k + \underbrace{\frac{\gamma}{2}(1 - \gamma)B\,t}_{\overset{\text{def}}{=}b} \right\}. \tag{54}$$

Note that (54) is a standard linear differential equation with integration factor $\psi(t) = \exp(n_k\gamma t)$ (for example, see [43], [44]). We get:

$$\bar{\rho}_k(t) = \left(\frac{b}{\gamma} \right) t + \left(\frac{a}{\gamma} - \frac{b}{n_k\gamma^2} \right) + \left(\rho_k(0) + \frac{b}{n_k\gamma^2} - \frac{a}{\gamma} \right) e^{-n_k\gamma t}, \tag{55}$$

so the expected densities grow linearly with time and the population is destined to conform to the external bias.

Figure 9 compares the theoretical prediction with simulations. Note that expected states cannot exceed $+1$, so after a finite time we have $s_x = +1$ for all x, which means that the whole population complies to the state imposed externally.
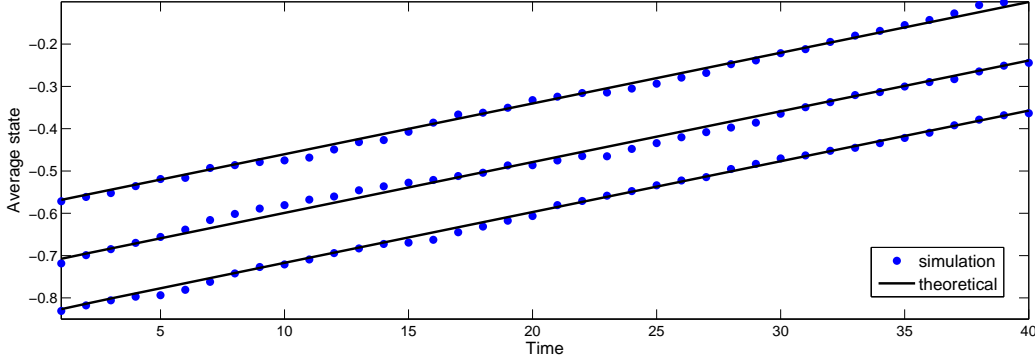

Fig. 9: m(t) with respect to time, for $\rho_0 = .08$ (bottom),0.14 (middle) and 0.21 (top) for $\gamma = 0.7$ and $B = 0.04$. The underlying graph is a Barabsi-Albert graph [37] with 200 nodes. Results are averaged over 500 Monte Carlo trials.

IV. THE EFFECT OF ANTICONFORMISTS

Now we seek the minimum number of nodes necessary to constantly adopt $s = -1$ in order to cancel off the external influence. Let us denote the number of stubborn neighbors of a (non-stubborn) node at site x by q_x, and denote the average number of stubborn neighbors among regular nodes by $\bar{q}$. Let the total number of stubborn nodes be Q and let μ and m be defined as above, only among ordinary nodes. Also let $\bar{z}$ be the average degree, only taking into account the ordinary nodes, not the stubborn ones.

The equation of motion at (non-stubborn) site x becomes

$$\dot{c}_x = \gamma \left\{ -c_x + \frac{\theta}{z_x + q_x} \left[-q_x + z_x \left(\frac{2\mu}{\bar{z}} - 1 \right) \right] + \frac{1-\theta}{N+Q}(Nm - Q) \right\} + (1-\gamma)B. \tag{56}$$

Let us define the constants

$$\begin{cases} \Lambda \overset{\text{def}}{=} \frac{1}{N} \sum_x \frac{q_x}{q_x + z_x} \\ \Omega \overset{\text{def}}{=} \frac{1}{N} \sum_x \frac{z_x q_x}{q_x + z_x} \end{cases}. \tag{57}$$

Then using (56) we get

$$\dot{m} = -\gamma \left(\frac{\theta + \bar{q}}{1 + \bar{q}} \right) m + \left(\frac{2\theta\gamma\Lambda}{\bar{z}} \right) \mu - \gamma\theta + \left[B(1 - \gamma) - \gamma\bar{q}\frac{1 - \theta}{1 + \bar{q}} \right]. \tag{58}$$

For $\mu(t)$ we have:

$$\dot{\mu} = \left[\frac{\gamma\bar{z}(1 - \theta)}{2(1 + \bar{q})} \right] m + \left(-\gamma + \frac{\gamma\theta\Omega}{\bar{z}} \right) \mu + \frac{\gamma\bar{z}}{2}(1 - \theta) + \frac{\bar{z}}{2} \left[(1 - \gamma)B - \frac{\gamma(1 - \theta)}{(1 + \bar{q})}\bar{q} \right]. \tag{59}$$

To cancel off the terms involving the external field, it suffices to have

$$\bar{q} = \frac{B(1 - \gamma)}{(1 - \theta)\gamma - B(1 - \gamma)}. \tag{60}$$

This is equivalent to

$$\gamma\bar{q}\frac{1 - \theta}{1 + \bar{q}} = B(1 - \gamma), \tag{61}$$

so the last terms in (58) and (59) vanish. This rules out the external excitation from the dynamics. Combining (58) and (59), the equation of motion for the average state becomes

$$\ddot{m} + \dot{m}\gamma \left(\frac{\theta + \bar{q}}{1 + \bar{q}} + 1 - \frac{\theta\Omega}{\bar{z}} \right) + m\gamma^2 \left[\frac{\theta + \bar{q}}{1 + \bar{q}} \left(1 - \frac{\theta\Omega}{\bar{z}} \right) - \theta\Lambda \left(\frac{1 - \theta}{1 + \bar{q}} \right) \right] = \gamma^2\theta \left(\Lambda(1 - \theta) - 1 + \frac{\theta\Omega}{\bar{z}} \right). \tag{62}$$

This is a second order linear differential equation. To analyze the roots, let us define

$$\begin{cases} K_1 \stackrel{\text{def}}{=} \dfrac{\theta + \bar{q}}{1 + \bar{q}} \\[2mm] K_2 \stackrel{\text{def}}{=} 1 - \dfrac{\theta\Omega}{\bar{z}} \\[2mm] K_3 \stackrel{\text{def}}{=} \theta\Lambda \left(\dfrac{1 - \theta}{1 + \bar{q}} \right) \end{cases}. \tag{63}$$

Then the roots of (62) are

$$\begin{cases} r_1 = \dfrac{-(K_1 + K_2) - \sqrt{(K_1 - K_2)^2 + 4K_3}}{2} \\[4mm] r_2 = \dfrac{-(K_1 + K_2) + \sqrt{(K_1 - K_2)^2 + 4K_3}}{2}. \end{cases} \tag{64}$$

It is clear that r_1 is negative. Now let us examine r_2. If $K_1 K_2 \geq K_3$, then r_2 is negative. This means that, if the following relationship holds:

$$(\theta + \bar{q}) \left(1 - \frac{\theta\Omega}{\bar{z}} \right) \geq \theta\Lambda(1 - \theta), \tag{65}$$

which is equivalent to

$$\Lambda \leq \left(\frac{1 - \frac{\theta\Omega}{\bar{z}}}{1 - \theta} \right) \left(\frac{\theta + \bar{q}}{\theta} \right), \tag{66}$$

then r_2 will be negative. First, note that

$$\Lambda = \frac{1}{N} \sum_x \frac{q_x}{q_x + z_x} \leq \frac{1}{N} \sum_x 1 = 1. \tag{67}$$

So Λ is less than or equal to unity. Thus, if we show that the right hand side of (66) is at least one, then (66) holds. To see this, first note that:

$$\Omega = \frac{1}{N} \sum_x \frac{z_x q_x}{q_x + z_x} \leq \frac{1}{N} \sum_x z_x = \bar{z}, \tag{68}$$

or equivalently,

$$\frac{\Omega}{\bar{z}} \leq 1. \tag{69}$$

Now let us re-write the right hand side of (66) for convenience. It is equal to:

$$\left(\frac{1 - \frac{\theta \Omega}{\bar{z}}}{1 - \theta} \right) \left(\frac{\theta + \bar{q}}{\theta} \right). \tag{70}$$

For the first factor we have:

$$\frac{\Omega}{\bar{z}} \leq 1 \implies \theta \frac{\Omega}{\bar{z}} \leq \theta. \tag{71}$$

Since $\theta \leq 1$, this leads to:

$$\left(\frac{1 - \frac{\theta \Omega}{\bar{z}}}{1 - \theta} \right) \geq 1. \tag{72}$$

For the second factor, we have:

$$\left(\frac{\theta + \bar{q}}{\theta} \right) = 1 + \frac{\bar{q}}{\theta} \geq 1. \tag{73}$$

So we have found that

$$\Lambda \leq 1 \leq \left(\frac{1 - \frac{\theta \Omega}{\bar{z}}}{1 - \theta} \right) \left(\frac{\theta + \bar{q}}{\theta} \right), \tag{74}$$

which means that (66) holds. Thus $r_2 \leq 0$.

As we have shown, both roots of (62) are negative, so the homogeneous answer decays, and the particular response prevails. We obtain an expression for the particular response by setting the time derivatives in (62) equal to zero. This gives the steady-state solution:

$$m(\infty) = \frac{\theta \left(\Lambda(1 - \theta) - 1 + \frac{\theta \Omega}{\bar{z}} \right)}{\left[\frac{\theta + \bar{q}}{1 + \bar{q}} \left(1 - \frac{\theta \Omega}{\bar{z}} \right) - \theta \Lambda \left(\frac{1 - \theta}{1 + \bar{q}} \right) \right]}. \tag{75}$$

The denominator is positive as we showed above. For the numerator, note that the following holds:

$$\Lambda \leq 1 \leq \left(\frac{1 - \frac{\theta \Omega}{\bar{z}}}{1 - \theta} \right), \tag{76}$$

or equivalently,

$$\Lambda(1 - \theta) \leq \left(1 - \frac{\theta \Omega}{\bar{z}} \right) \tag{77}$$

So the numerator of (75) is negative, so $m(\infty)$ is negative. Remember that the exogenous stimulus was trying to bias the states towads $s = +1$ and the anti-conformists were trying to pull the states towards $s = -1$. Equation (75) states that the average steady-state opinion of the whole population is negative, which is in favor of the stubborn nodes.

We conclude that the distribution of stubborn nodes controls the equilibrium bias through Λ and Ω, given that (60) holds. Figure 10 shows the simulation results and convergence to what (75) predicts.
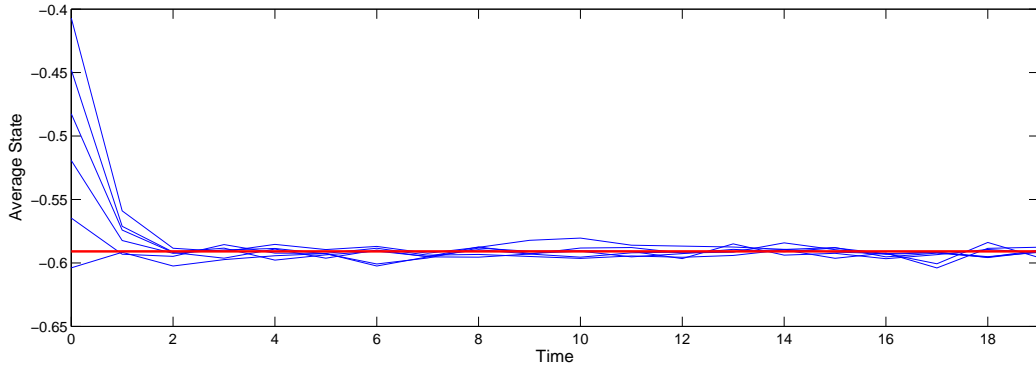

Fig. 10: m(t) with respect to time, for different values of ρ between zero and 0.9 (uniform), the red line is the prediction of (75). The underlying graph is random with 500 nodes. The results are averaged over 100 Monte Carlo trials.

Now let us examine a special case. Assume that q_k only depends on k, which means that nodes with the same degree have equal number of stubborn neighbors. Let the fraction of stubborn neighbors for nodes of each degree to be the same; i.e., there is some $\alpha \leq 1$ that $q_k = \alpha k$ for all k. This means

$$\alpha = \frac{\frac{B}{\bar{z}}(1-\gamma)}{(1-\theta)\gamma - B(1-\gamma)}. \tag{78}$$

For the parameters in (57) we have:

$$\begin{cases} \Lambda = \frac{\alpha}{\alpha+1} \\ \Omega = \left(\frac{\alpha}{\alpha+1}\right)\bar{z} \end{cases}. \tag{79}$$

The final average state becomes:

$$m(\infty) = \frac{\frac{-\theta}{\alpha+1}}{\frac{\theta+\alpha\bar{z}+\alpha^2\bar{z}(1-\theta)}{(1+\alpha\bar{z})(1+\alpha)}} < 0, \tag{80}$$

which is negative, meaning that the bias leans towards the stubborn nodes. Note that for the anticonformists to be able to succeed, the necessary condition is

$$\gamma > \frac{2B}{2B+1}, \tag{81}$$

which means that the population must care less than some threshold for the inflicted external bias.

V. SUMMARY AND DISCUSSION

We considered the problem of binary states, or opinions, among nodes that are connected through a network of arbitrary degree distribution. We modelled the social influence by two distinct parts. Each node finds the fraction of nodes who are adjacent to it and disagree with it, and also the fraction of the nodes in the whole network that disagree with it. It then takes a convex combination of those fractions, to find a probability, and at each timestep, flips its opinion with this probability. We solved for the probability of consensus on either state, and the expected time to reach consensus, as a function of initial conditions. We also found the time evolution of the average state over the whole population.

Then we added an external influence to the model, that tries to bias all nodes towards one of the states. We solved for the average state of the system in time.

To cancel off the effect of the exogenous bias, we introduced stubborn nodes. These nodes constantly oppose the external bias. We found the equilibrium average state of the system, and the necessary conditions so that the stubborn nodes can succeed in defeating the external field.

Possible extensions of the model include using a network topology with given degree correlations, and using it to improve the mean field assumption. Also, it will be more realistic, albeit analytically somewhat formidable, if the influence between each pair of nodes was a function of the distance between them. Alternatively, each node, when accounting for the state of other nodes, can assign different weights to them, that can be functions of other types of centralities. In this case, the more central a node is, the more influence it exerts on the nodes it interacts with. Upon existence of the external field, nodes can be given different levels of exposure, or different degrees of resistance against it. This means that, stubbornness can be a quality that exists in all nodes, but to various degrees.

Another useful modification, especially for marketing purposes, is as follows. Suppose $s = -1$ means that the node have not bought some product, or have not watched some movie. Then, one can modify the model so that nodes could not go back once they adopted $s = +1$ (so-called "susceptible-infected" models in the epidemiology literature). This means that, for example, once they saw the movie, they cannot un-see it. This will render the model readily testable against empirical data.

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